Name ______________

Asymptotes of Rational Functions

x=a

Vertical Asymptotes

Definition: The line $x = a$ is a vertical asymptote of the graph of $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as x approaches "a" from either the left or the right.

→ Happen at the value(s) for which the domain is undefined.

- → Set the denominator=0 and solve for x.
- → (Note that these x-values are only vertical asymptotes if, when x is plugged back in, the numerator $\neq 0$)

→ The graph of a rational function will NEVER cross a vertical asymptote.

EXAMPLES:

Find the vertical asymptotes for the following functions:

1. $f(x) = \dfrac{1}{x+2} \neq 0$

   $x = -2$

2. $f(x) = \dfrac{x-1}{(x+2)(x+1)}$

   $x = -2 \quad x = -1$

3. $f(x) = \dfrac{x}{x^2+1} \neq 0$

   $x^2 = -1$

   No vertical asymptotes

4. $f(x) = \dfrac{2x^2-3x-1}{x-2}$

   $x = 2$

5. $f(x) = \dfrac{2x^2+7x-4}{x^2+x-2}$

   $(x-1)(x+2)$

   $x = 1 \quad x = -2$

6. $f(x) = \dfrac{5x}{x^2-4}$

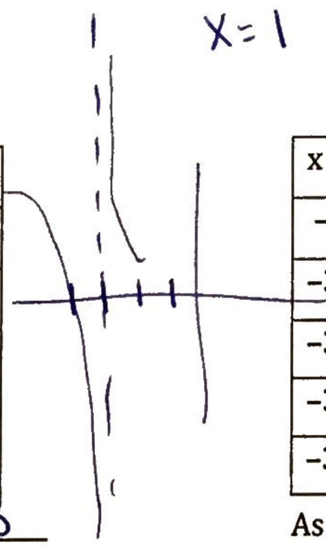
   $(x-2)(x+2)$

   $x = 2, -2$

7.

x	f(x)
-2	1
-2.5	2.222
-2.9	11.837
-2.99	119.84
-2.999	1199.8

As x approaches -3 from the right, f(x) approaches ∞.



x	f(x)
-4	-1.333
-3.5	-2.545
-3.1	-12.16
-3.01	-120.2
-3.001	-1200

As x approaches -3 from the left, f(x) approaches $-\infty$.

V.A. $x = -3$

Horizontal Asymptotes

Definition: The line $y = b$ is a horizontal asymptote of the graph of $f(x)$ if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

→ If the degree of the numerator is less than the degree of the denominator (in other words, the function is **proper**), the horizontal asymptote is $y = 0$ #1,2

→ If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the ratio of the leading coefficients. #3,4

→ If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote. #5,6

→ Horizontal asymptotes describe the <u>end</u> behavior of a function. A graph MAY (or might not) cross a horizontal asymptote between the ends, but the graph must level off at one or both ends.

EXAMPLES:

Find the horizontal asymptotes for the following functions:

1. $f(x) = \frac{4}{x^2+1}$

$y=0$

2. $f(x) = \frac{2x}{3x^2+1}$

$y=0$

3. $f(x) = \frac{2x+1}{1x+1}$

$y = \frac{2}{1} = 2$

4. $f(x) = \frac{5x^2+1}{2x^2-8}$

$y = \frac{5}{2}$

5. $f(x) = \frac{3x^3 - \cancel{5x^2 + 4x - 5}}{3x+1}$

no horiz. asympt.

6. $f(x) = \frac{x^2-9}{x+2}$

no horiz. asymptote

7.

x	f(x)
−1	1
−10	−0.125
−100	−0.0102
−1000	−0.001

$y \to$ 0 as $x \to$ $-\infty$

x	f(x)
1	0.333
10	0.0833
100	0.0098
1000	0.0009

$y \to$ 0 as $x \to$ ∞

$y=0$

Oblique/Slant Asymptotes

The graph of a rational function has a slant asymptote if the degree of the numerator is exactly one more than the degree of the denominator.

→ Use polynomial long division to find the slant asymptotes. (Ignore the remainder).

→ You cannot have BOTH a slant asymptote and a horizontal asymptote.

EXAMPLES:

Find the slant/oblique asymptotes for the following functions:

1. $f(x)=\frac{3x^4-x^2}{x^3-x^2+1} = \boxed{3x+3} + \cancel{\frac{2x^2-3x-3}{x^3-x^2+1}}$

$$\begin{array}{r} 3x+3 \\ x^3-x^2+1 \overline{)\, 3x^4+0x^3-x^2} \\ -3x^4+3x^3-3x \\ \hline 3x^3-x^2-3x \\ -3x^3+3x^2-3 \\ \hline 2x^2-3x-3 \end{array}$$

$3x(x^3-x^2+1)$

$-(3x^4-3x^3+3x)$

$3(x^3-x^2+1)$

$-(3x^3-3x^2+3)$

$y=3x+3$

2. $f(x)=\frac{x^2-x-2}{x-1} = \boxed{x} \cancel{\frac{-2}{x-1}}$

$$\begin{array}{r} x \\ x-1 \overline{)\, x^2-x-2} \\ -x^2+x \\ \hline -2 \end{array}$$

$x(x-1)$

$-(x^2-x)$

$y=x$

3. $f(x)=\frac{x^2-2}{x-1} = \boxed{x+1} \cancel{-\frac{1}{x-1}}$

$$\begin{array}{r} x+1 \\ x-1 \overline{)\, x^2-2} \\ -x^2+x \\ \hline x-2 \\ -x+1 \\ \hline -1 \end{array}$$

$x(x-1)$

$-(x^2-x)$

$1(x-1)$

$-(x-1)$

$y=x+1$

4. $f(x)=\frac{8x^3-x^2+3x+2}{4x^2-1} = \boxed{2x-\frac{1}{4}} + \cancel{\frac{5x+\frac{7}{4}}{4x^2-1}}$

$$\begin{array}{r} 2x-\frac{1}{4} \\ 4x^2-1 \overline{)\, 8x^3-x^2+3x+2} \\ -8x^3 \quad +2x \\ \hline -x^2+5x+2 \\ x^2 \quad -\frac{1}{4} \\ \hline 5x+\frac{7}{4} \end{array}$$

$2x(4x^2-1)$

$-(8x^3-2x)$

$-\frac{1}{4}(4x^2-1)$

$-(-x^2+\frac{1}{4})$

$y=2x-\frac{1}{4}$

MIXED PRACTICE:

Find ALL asymptotes (vertical, horizontal, and/or slant) of the given functions.

1. $f(x)=\frac{3x-2}{(x-1)(x+4)}$ → x^2. . .

vertical: $x=1,-4$

horizontal: $y=0$

2. $f(x)=\frac{1x^2-16}{1x^2+4x-21}$ $\frac{(x-4)(x+4)}{(x+7)(x-3)}$

vertical: $x=-7,3$

horizontal $y=\frac{1}{1}=1$